# SMALL PERTURBATIONS OF A SPHERICAL FLUID LAYER

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UDC 532.51

The evolution of small perturbations of an ideal incompressible fluid is investigated in the presence of surface tension forces, when the principal motion is unstable motion of a spherical layer with free boundaries.

1. Description of the Basic Motion. At time $t = 0$ let an ideal incompressible fluid occupy a spherical layer $0 < r_{10} \leqslant r \leqslant r_{20}$ with an assigned velocity field $u_r = T_0/r^2$, $u_\theta = u_\varphi = 0$. Here $r_{10}$, $r_{20}$, and $T_0$ are constants, and $u_r$, $u_\theta$, $u_\varphi$ are the velocity components in spherical coordinate system $(r, \theta, \varphi)$. We assume that at $t > 0$ the motion is also spherically symmetric: $u_r = u_r(r, t)$, $u_\theta = u = 0$. Taking into account surface tension forces and the pressure drop at the internal and external boundaries, the motion of the layer is then described by the equations

$$p = \rho_0 \alpha(t) + \rho_0 \left(\frac{1}{r}\frac{dT}{dt} - \frac{T^2}{2r^4}\right), \quad u_r = \frac{T(t)}{r^2}, \quad r_1(t) \leqslant r \leqslant r_2(t); \tag{1.1}$$

$$p(r_2(t), t) - p(r_1(t), t) = P(t) + \frac{2\sigma_1}{r_1(t)} + \frac{2\sigma_2}{r_2(t)}; \tag{1.2}$$

$$\frac{dr_1}{dt} = \frac{T(t)}{r_1^2}, \quad \frac{dr_2}{dt} = \frac{T(t)}{r_2^2}; \tag{1.3}$$

$$T(0) = T_0, \; r_1(0) = r_{10}, \; r_2(0) = r_{20}, \tag{1.4}$$

where $r_1(t)$, $r_2(t)$ are the radii; $\sigma_1$, $\sigma_2$, surface tension coefficients of the inner and outer layer boundaries, respectively (generally speaking, $\sigma_1 \neq \sigma_2$, since the inner and outer layers can consist of different media); $P(t)$, assigned pressure drop; $\rho_0$, fluid density; and $\alpha(t)$, an arbitrary function. The expression for the radial velocity $u_r$ in (1.1) is found from the equation of continuity, and the pressure $p$ is found by integrating the equations of motion. Relations (1.2), (1.3) are the dynamic and kinematic conditions at the free layer boundaries $r = r_1(t)$, $r = r_2(t)$, and with the initial conditions (1.4) they determine the unknown functions $T(t)$, $r_1(t)$, $r_2(t)$. The quantities $r_1(t)$ and $r_2(t)$ are related by the conservation law of the layer volume V

$$r_2^3 - r_1^3 = 3V/4\pi = r_{20}^3 - r_{10}^3,$$

which follows directly from (1.3), (1.4).

For $P = 0$, $\sigma_1 = \sigma_2 = 0$ approximate expressions were obtained [1] for $r_1(t)$, $r_2(t)$, $T(t)$ in the cases of thin ($h/r_{10}^3 \ll 1$, $h = r_{20}^3 - r_{10}^3$) and thick ($r_{10}^3/h \ll 1$) layers. Thus, for a thin compressible layer

$$r_1(t) = r_{10}\left[1 - a_1 t - \frac{h}{6r_{10}^3}\frac{1}{(1-a_1 t)^2} + \frac{h}{6r_{10}^3}\right], \quad a_1 = \frac{T_0}{r_{10}}\left(\frac{r_{10}^{-1} - r_{20}^{-1}}{h}\right)^{1/2},$$

$$r_2(t) = r_{20}\left[1 - a_2 t + \frac{h}{6r_{20}^3}\frac{1}{(1-a_2 t)^2} - \frac{h}{6r_{20}^3}\right], \quad a_2 = \frac{r_{10}}{r_{20}} a_1, \tag{1.5}$$

$$T(t) = -a_1 r_{10}^3\left[(1-a_1 t)^2 + \frac{h}{3r_{10}^3}\right].$$

The equations for a thick layer are more cumbersome.

This problem was considered in [2] as an example of studying the instability of motion of a fluid with free boundaries in the case $\sigma_1 = \sigma_2 = 0$. The motion of a spherical layer under the action of capillary forces only ($\sigma_1 = \sigma_2 \neq 0$, $P = 0$, $T_0 = 0$) was considered in [3],

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Krasnoyarsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 110-117, January-February, 1981. Original article submitted November 11, 1979.

0021-8944/81/2201-0091$07.50  

where curves were provided of the dimensionless collapse time of the layer as a function of the ratio $r_{10}/r_{20}$.

To determine the law of motion, we introduce the new function $\gamma(t)$

$$\gamma(t)=3\int_0^t T(t)\,dt, \qquad \gamma(0)=0. \tag{1.6}$$

From (1.3), (1.4), we find

$$r_1=(r_{10}^3+\gamma(t))^{1/3}, \quad r_2=(r_{20}^3+\gamma(t))^{1/3}. \tag{1.7}$$

Substituting the expression for the pressure into (1.2) and using (1.6), (1.7), we arrive at the Cauchy problem for the function $\gamma(t)$

$$\begin{gathered}[(r_{20}^3+\gamma)^{-1/3}-(r_{10}^3+\gamma)^{-1/3}]\gamma''-\frac{1}{6}[(r_{20}^3+\gamma)^{-4/3}-\\ -(r_{10}^3+\gamma)^{-4/3}](\gamma')^2=\frac{6\sigma_1}{\rho_0}(r_{10}^3+\gamma)^{-1/3}+\frac{6\sigma_2}{\rho_0}(r_{20}^3+\gamma)^{-1/3}+\frac{3P(t)}{\rho_0};\end{gathered} \tag{1.8}$$

$$\gamma(0)=0, \ \gamma'(0)=3T_0. \tag{1.9}$$

We restrict ourselves to the case of a constant pressure drop $P(t) = P_0 = \text{const}$. Equation (1.8) is integrated:

$$\begin{gathered}\left(\frac{d\gamma}{dt}\right)^2=\frac{6}{\rho_0}\frac{D-3\sigma_1(r_{10}^3+\gamma)^{2/3}-3\sigma_2(r_{20}^3+\gamma)^{2/3}-P_0\gamma}{\left[(r_{10}^3+\gamma)^{-1/3}-(r_{20}^3+\gamma)^{-1/3}\right]},\\ D=\frac{3}{2}\rho_0T_0^2(r_{10}^{-1}-r_{20}^{-1})+3\sigma_1r_{10}^2+3\sigma_2r_{20}^2.\end{gathered} \tag{1.10}$$

It is clear that for $\sigma_1 \neq 0$, $\sigma_2 \neq 0$ the function $\gamma(t)$ increases monotonically for $T_0 > 0$ until a certain moment of time $t = t_1$, and then decreases and $\gamma(t) \to -r_{10}^3$ when $t \to t_2 < \infty$. In this case the layer initially diverges and reaches a maximum radius $r_{1*} = (r_{10}^3 + \gamma_*(t_1))^{1/3}$, at moment $t = t_1$, where $\gamma_*$ is defined by the equality

$$3\sigma_1(r_{10}^3+\gamma_*)^{2/3}+3\sigma_2(r_{20}^3+\gamma_*)^{2/3}+P_0\gamma_*=D, \tag{1.11}$$

and then converges to the center under the action of capillary forces after time $t = t_2 - t_1$. For $T_0 \leqslant 0$ the layer immediately converges to the center after a finite time.

If $\sigma_1 = \sigma_2 = 0$, $P_0 = 0$ (inertial layer motion) and $T_0 > 0$, $\gamma(t)$ increases monotonically, $\gamma \to \infty$, $t \to \infty$, the layer diverges to infinity, and its width tends to zero: $r_2(t) - r_1(t) \sim (h\gamma^{-2/3})/3$. For $T_0 < 0$ there occurs a collapse at a finite time $t_3$, defined by the equation

$$t_3=\sqrt{\frac{\rho_0}{6D}}\int_{-r_{10}^3}^{0}\sqrt{(r_{10}^3+\gamma)^{-1/3}-(r_{20}^3+\gamma)^{-1/3}}\,d\gamma. \tag{1.12}$$

The motion under the action of a constant pressure drop $P_0$ only can be considered, when the parameter is $P|_{r=r_2(t)} = P_0$, and the internal sphere $r = r_1(t)$ is free. In this case it is sufficient to put $T_0 = 0$, $\sigma_1 = \sigma_2 = 0$ in Eqs. (1.2), (1.10), and (1.11).

Knowing the function $\gamma(t)$, one uniquely finds $T(t)$, $r_1(t)$, and $r_2(t)$ according to (1.6), (1.7), and this description of the basic motion is complete, it being a potential motion with potential $T(t)/r$.

For what follows it is required to write the basic motion (1.1) in Lagrange coordinates $\xi = (\xi_1, \xi_2, \xi_3)$ [2]:

$$\mathbf{x}=m(\rho,t)\,\xi, \ \mathbf{u}=\mathbf{x}_t=m_t\xi \ (\rho=|\xi|); \tag{1.13}$$

$$p/\rho_0=\alpha(t)+\frac{\gamma''}{3}(\rho^3+\gamma)^{-1/3}-\frac{(\gamma')^2}{18}(\rho^3+\gamma)^{-4/3}, \tag{1.14}$$

where

$$m(\rho,t)=\rho^{-1}(\rho^3+\gamma)^{1/3}, \tag{1.15}$$

and $\gamma(t)$ satisfies Eq. (1.10).

In concluding this point, we provide the value of the radius $r_{**}$ for which the pressure is maximum:

$$r_{**}=[2(\gamma')^2/3\gamma'']^{1/3}.$$

It can be shown that $r_1(t) < r_{**}(t) < r_2(t)$.

2. Small Perturbation Equations of a Spherical Layer. As shown in [4], the evolution problem of small perturbations of an arbitrary potential flow of an ideal incompressible fluid with account of capillary forces can be reduced to the following equations:

$$\operatorname{div} M^{-1}M^{*-1}\nabla\Phi=0,\ \xi\in\Omega,\ t\geqslant 0; \tag{2.1}$$

$$\rho_0\Phi_t=\left[\frac{\partial p}{\partial n_{\Gamma_t}}+\sigma\left(\frac{1}{R_1^2}+\frac{1}{R_2^2}\right)\right]R+\sigma\bar{\Delta}_\Gamma(t)R,\quad \xi\in\Gamma,\ t>0; \tag{2.2}$$

$$R=\frac{|\nabla f|}{|M^{*-1}\nabla f|}\mathbf{n}\cdot\left(1+\int_0^t M^{-1}M^{*-1}\nabla\Phi dt\right),\quad \xi\in\Gamma,\ t\geqslant 0; \tag{2.3}$$

$$\Phi|_{t=0}=\Phi_0,\ \Delta\Phi_0=0. \tag{2.4}$$

In Eqs. (2.1)-(2.4) M is the matrix of the Jacobi mapping $\xi\to\mathbf{x}(\xi, t)$ of the initial region $\Omega$ to the flow region $\Omega_t$ for $t > 0$ with elements $M_{ik} = \partial x_i/\partial\xi_k$ (i, k = 1, 2, 3); $M^*$, adjoint matrix; $\Gamma$, boundary of $\Omega$; $f(\xi) = 0$, its equation; $\mathbf{n}$, normal to $\Gamma$; $\Gamma_t$, boundary of $\Omega_t$; $R_1$ and $R_2$, principal radii of curvature of its normal cross sections; and $\partial p/\partial n_{\Gamma_t}$, normal derivatives of the pressure p with respect to $\Gamma_t$. Besides, $\bar{\Delta}_\Gamma(t)$ is the Laplace–Beltrami operator with coefficients $E = |M\xi_\alpha|^2$, $G = |M\xi_\beta|^2$, $F = M\xi_\alpha\cdot M\xi_\beta$, where $(\alpha, \beta)\to\xi(\alpha, \beta)$ is some regular parametrization of the boundary $\Gamma$, and $\Gamma(\xi)$ $(\xi\in\Gamma)$ is the displacement vector of boundary points, characterizing the initial perturbation of the flow region.

Knowing the function $\Phi(\xi, t)$, the pressure and velocity perturbations are determined in the form

$$\tilde{p}=-\rho_0\Phi_t,\ \tilde{\mathbf{u}}=M^{*-1}\nabla\Phi. \tag{2.5}$$

The function $R(\xi, t)$, $\xi\in\Gamma$, is the derivation of the free boundary in the perturbed motion from the free boundary [4]. In stability problems one is usually also interested in the behavior of $R(\xi, t)$, $t\to\infty$. Therefore, the stability of some potential flow is the problem of asymptotic behavior of the solution of problems (2.1)-(2.4).

The solution of problem (2.1)-(2.4) for $\sigma > 0$ always exists and is unique if the principal motion and the boundary $\Gamma$ are sufficiently smooth [4]. If also $\sigma = 0$ then, as shown by corresponding examples [5], for $\partial p/\partial n_{\Gamma_t} > 0$ the problem is incorrectly stated according to Hadamard. The surface tension is a regularizing factor.

In our case the region $\Omega$ is the spherical layer $r_{10} < \rho < r_{20}$ with boundaries $\Gamma_1(\rho = r_{10})$ and $\Gamma_2(\rho = r_{20})$, therefore it is natural to put $\alpha = \theta$, $\beta = \varphi$, $-\pi\leqslant\theta\leqslant\pi$, $0\leqslant\varphi\leqslant 2\pi$, $\xi(\theta, \varphi) = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$.

We use the following equations for the principal motion (1.13)-(1.15) [2]:

$$M=M^*=m\mathscr{E}-\frac{m^3-1}{m^2}Q,\quad M^{-1}=\frac{1}{m}\mathscr{E}+\frac{m^3-1}{m}Q,$$

where $\mathscr{E}$ is the unit matrix, and Q is the matrix with components $Q_{ik} = \xi_i\xi_k/\rho^2$ (i, k = 1, 2, 3), with $Q^2 = Q$, $Q\xi = \xi$, $Q\nabla\Phi = \Phi_\rho\xi/\rho$. By means of these equations, Eq. (2.1) transforms to the form

$$m^4\left[\Phi_{\rho\rho}+\frac{2}{\rho}\frac{\rho^3-\gamma}{\rho^3+\gamma}\Phi_\rho+\frac{\rho^6}{(\rho^3+\gamma)^2}\Delta_S\Phi\right]=0, \tag{2.6}$$

where $\Delta_S$ is the surface part of the Laplace operator

$$\Delta_S\Phi=\frac{1}{\rho^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}\right].$$

We perform a transformation of the boundary condition (2.2), where one must put $\sigma = \sigma_1$ for $\rho = r_{10}$, and $\sigma = \sigma_2$ for $\rho = r_{20}$. Denoting $m_j(t) = m(r_{j0}, t)$ (j = 1, 2), we find $E = (r_{j0}m_j)^2$, $G = (r_{j0}m_j\sin\theta)^2$, $F = 0$. Therefore, $\bar{\Delta}_\Gamma(t)R = \Delta_S R/m_j^2$. Since the equation for the free boundary $\Gamma_j$ is $f = \rho - r_{j0} = 0$ $(j = 1, 2)$, $1/R_1^2 + 1/R_2^2 = 2/r_{j0}^2 m_j^2$, $|\nabla f|/|M^{*-1}\times \nabla f| = 1/m_j^2$, $n_1 = -\xi/r_{10}$, $n_2 = \xi/r_{20}$, we obtain from (1.1), (1.14), (2.3)

$$\frac{\partial p}{\partial n_{\Gamma_t}}\Big|_{r=r_1(t)} = \rho_0 r_{10} m_{1tt}, \quad \frac{\partial p}{\partial n_{\Gamma_t}}\Big|_{r=r_2(t)} = -\rho_0 r_{20} m_{2tt}; \tag{2.7}$$

$$R\,|_{r=r_1(t)} = \frac{1}{m_1^2}\left(a - \int_0^t m_1^4 \Phi_\rho dt\right), \quad R\,|_{r=r_2(t)} = \frac{1}{m_2^2}\left(b + \int_0^t m_2^4 \Phi_\rho dt\right), \tag{2.8}$$

$$a = \mathbf{n}_1 \cdot \mathbf{l}\,|_{\Gamma_1}, \quad b = \mathbf{n}_2 \cdot \mathbf{l}\,|_{\Gamma_2}.$$

The boundary condition (2.2) gives now two relations

$$\rho_0 \Phi_t + \frac{\sigma_1}{m_1^4}\int_0^t m_1^4 \Delta_S \Phi_\rho dt + \frac{1}{m_1^2}\left(\rho_0 r_{10} m_{1tt} + \frac{2\sigma_1}{r_{10}^2 m_1^2}\right)\int_0^t m_1^4 \Phi_\rho dt = \frac{\sigma_1}{m_1^4}\Delta_S a + \frac{1}{m_1^2}\left(\rho_0 r_{10} m_{1tt} + \frac{2\sigma_1}{r_{10}^2 m_1^2}\right) a, \quad \rho = r_{10}; \tag{2.9}$$

$$\rho_0 \Phi_t - \frac{\sigma_2}{m_2^4}\int_0^t m_2^4 \Delta_S \Phi_\rho dt - \frac{1}{m_2^2}\left(\frac{2\sigma_2}{r_{20}^2 m_2^2} - \rho_0 r_{20} m_{2tt}\right)\int_0^t m_2^4 \Phi_\rho dt = \frac{\sigma_2}{m_2^4}\Delta_S b + \frac{1}{m_2^2}\left(\frac{2\sigma_2}{r_{20}^2 m_2^2} - \rho_0 r_{20} m_{2tt}\right) b, \quad \rho = r_{20}. \tag{2.10}$$

Thus, to analyze the behavior of small perturbations of the spherical layer, it is required to find the function $\Phi(\rho, \theta, \varphi, t)$ as the solution of the initial-boundary-value problem (2.6), (2.9), (2.10), (2.4).

3. Inertial Motion of the Layer. In this case $\sigma_1 = \sigma_2 = 0$, and the layer either diverges to infinity, or converges to the center after a time $t_*$, determined by Eq. (1.12).

It must be noted that in this case, problem (2.6), (2.9), (2.10), (2.4) is correctly stated, since by (1.3), (1.13), (2.7)

$$\frac{\partial p}{\partial n_{\Gamma_t}}\Big|_{r=r_1(t)} = \frac{\rho_0 r_2 T^2}{2 r_1^5 (r_1 - r_2)}(\lambda - 1)^2(\lambda^2 + 2\lambda + 3) < 0, \quad \lambda = \frac{r_1}{r_2},$$

$$\frac{\partial p}{\partial n_{\Gamma_t}}\Big|_{r=r_2(t)} = \frac{\rho_0 T^2}{2 r_2 (r_1 - r_2) r_1^3}(\lambda - 1)^2(3\lambda^2 + 2\lambda + 1) < 0$$

for the accurate solution. It can be shown, however, that for approximation (1.5) $\partial p/\partial n_{\Gamma_t} = \rho_0 r_{10}^{-2}\alpha_1^2 h(1 - \alpha_1 t)^{-4} > 0$, $r = r_1(t)$, and the linearized problem is incorrectly stated according to Hadamard at this approximation.

We proceed now to analyzing the small perturbation equations. Since in Eqs. (2.6), (2.9), (2.10) the variables $(\rho, t)$, $\theta$, $\varphi$ can be separated, we put

$$\Phi(\rho, \theta, \varphi, t) = \sum_{n=1}^{\infty}\sum_{k=0}^{n} A_{nk}(\rho, t) P_{nk}(\cos\theta) e^{ik\varphi}, \quad i = \sqrt{-1}, \tag{3.1}$$

where $P_{nk}(\cos\theta)$ are the associated Legendre functions.

Restricting ourselves to a single harmonic and denoting for brevity $A(\rho, t) = A_{nk}(\rho, t)$, we obtain from (2.6)

$$A_{\rho\rho} + \frac{2}{\rho}\frac{\rho^3 - \gamma}{\rho^3 + \gamma} A_\rho - \frac{\rho^4 n(n+1)}{(\rho^3 + \gamma)^2} A = 0 \tag{3.2}$$

which is an equation for $A(\rho, t)$. The general solution of this equation is represented in the form

$$A = C_1(t)(\rho^3 + \gamma)^{-n/3} + C_2(t)(\rho^3 + \gamma)^{\frac{n+1}{3}} \tag{3.3}$$

with arbitrary functions $C_1(t)$, $C_2(t)$. The boundary conditions generate the following relations:

$$\left(\frac{m_j^2}{m_{jtt}} A_t'\right)_t' + m_j^4 r_{j0} A_\rho' = 0, \quad \rho = r_{j0} \quad (j = 1, 2). \tag{3.4}$$

We transform to the independent variable $\tau(t) = r_{10}^3 + \gamma(t)$ instead of $t$. Due to (1.10), where one must put $\sigma_1 = \sigma_2 = 0$, $P_0 = 0$, the relation between $\tau$ and $t$ is one-to-one for $t \geqslant 0$. We denote $A_1 = A(r_{10}, \tau)$, $A_2 = A(r_{20}, \tau)$ and express functions $C_1(t)$, $C_2(t)$ by means of equality (3.3) in terms of $A_1(\tau)$, $A_2(\tau)$. Besides, we find $A_\rho$ from (3.3) for $\rho = r_{10}, r_{20}$. As a result of these transformations, one finally obtains a system of second-order equations in the functions $A_1(\tau)$, $A_2(\tau)$

$$\left(\frac{m_1^2}{m_{1tt}} A'_{1\tau}\right)'_\tau g + \frac{m_1^4 r_{10}^3}{\Delta(\tau)} \left\{\left[(2n+1)\tau^{\frac{n-2}{3}}(h+\tau)^{n/3}\right] A_2 - \left[(n+1)\tau^{\frac{2n-2}{3}} + n\tau^{-1}(h+\tau)^{\frac{2n+1}{3}}\right] A_1\right\} = 0; \quad (3.5)$$

$$\left(\frac{m_2^2}{m_{2tt}} A'_{2\tau}\right)'_\tau g + \frac{m_2^4 r_{20}^3}{\Delta(\tau)} \left\{\left[(n+1)(h+\tau)^{\frac{2n-2}{3}} + n\tau^{\frac{2n+1}{3}}(h+\tau)^{-1}\right] A_2 - \left[(2n+1)\tau^{n/3}(h+\tau)^{\frac{n-2}{3}}\right] A_1\right\} = 0, \quad (3.6)$$

where we put

$$g(\tau) = \sqrt{\frac{6D}{\rho_0}}\,[\tau^{-1/3} - (h+\tau)^{-1/3}]^{-1/2}; \quad \Delta(\tau) = (h+\tau)^{\frac{2n+1}{3}} - \tau^{\frac{2n+1}{3}}. \quad (3.7)$$

To construct the perturbed motion, we supplement the system by the initial conditions

$$A_1 = A_{10}, \quad A_2 = A_{20}, \quad A'_{1\tau} = \frac{r_{10} m_{1tt}(0)}{g(r_{10}^3)} a_{nk}, \quad A'_{2\tau} = \frac{r_{20} m_{2tt}(0)}{g(r_{10}^3)} b_{nk} \quad (3.8)$$

for $\tau = r_{10}^3$. Here the function $g(\tau)$ is given by Eq. (3.7), and the constants $a_{nk}$, $b_{nk}$ are the Fourier series corfficients in spherical harmonics of the initial deviations of the layer surface $a(\theta, \varphi)$, $b(\theta, \varphi)$. Restricting ourselves primarily to a single harmonic, we find from (2.8)-(2.10) (we recall that $\sigma_1 = \sigma_2 = 0$)

$$R_{nk} = -\frac{1}{r_{j0} m_{jtt}} g(\tau) A'_{j\tau} P_{nk}(\cos\theta) e^{ik\varphi} \qquad (j = 1, 2). \quad (3.9)$$

Using the explicit expressions for $m_j(\tau)$, $g(\tau)$, it can be shown that the coefficients of system (3.5), (3.6) have power singularities for $\tau \to \infty$ (diverging layer), and $\tau \to 0$ (converging layer). For example, for $\tau \to \infty$ the asymptotic behavior of $A_1(\tau)$, $A_2(\tau)$ is the same as that of the functions $\bar{A}_1$, $\bar{A}_2$, satisfying the system

$$(\tau^{8/3}\bar{A}'_{1\tau})'_\tau + \frac{1}{3}\tau^{2/3}(\bar{A}_1 - \bar{A}_2) = 0, \quad (\tau^{8/3}\bar{A}'_{2\tau})'_\tau - \frac{1}{3}\tau^{2/3}(\bar{A}_1 - \bar{A}_2) = 0.$$

The latter system is easily solved, and for the deviation of the free boundary from (3.9) we find outside the bands $|\theta| < \varepsilon$, $|\pi - \theta| < \varepsilon$ ($\varepsilon > 0$ is fixed)

$$R_{nk}\big|_{r=r_j(t)} \sim d_{nk}\tau^{1/3} P_{nk}(\cos\theta) e^{ik\varphi} \quad (j = 1, 2), \; d_{nk} = \text{const}, \quad (3.10)$$

i.e., both surfaces are unstable under expansion.

Under compression $\tau \to 0$, and it can be shown that $A_2 \sim d_1 + d_2\tau^{1/2}$ for $n = 1$, $A_2 \sim d_1 + d_2\tau^{1/6}$ for $n \geqslant 2$ ($d_1$ and $d_2$ are constants), and $A_1 \sim \tau^{-(n+2)/3}$ for $n \geqslant 1$. Using (3.9), (3.7), we find outside the bands $|\theta| < \varepsilon$, $|\pi - \theta| < \varepsilon$

$$R_{nk}\big|_{r=r_1(t)} \sim d_{nk,1}\tau^{-(2n+1)/6} P_{nk}(\cos\theta) e^{ik\varphi}, \quad d_{nk,1} = \text{const}; \quad (3.11)$$

$$R_{nk}\big|_{r=r_2(t)} \sim d_{nk,2}\tau^{2/3} P_{nk}(\cos\theta) e^{ik\varphi}, \quad d_{nk,2} = \text{const}, \quad (3.12)$$

i.e., for layer compression the external surface is stable, and the internal is not.

Consider the behavior of $R_{nk}$ for $\tau \to \infty$ ($\tau \to 0$), $\theta \to 0$ (the analysis of $\theta \to \pi$ is similarly performed). If $k \geqslant 1$ and $|\theta|^k \tau^{1/3 - \delta} = \text{const}$, $1/3 \geqslant \delta \geqslant 0$, we have from (3.11) $R_{nk} \sim \tau^{-\delta}$, $\tau \to 0$. We conclude from this analysis that there exist sharp stability bands of the layer boundaries near the poles $\theta = 0$, $\theta = \pi$. Similar stability bands also exist near the node lines of the associated Legendre functions.

Similarly restricting ourselves to a single harmonic in (2.5), we derive for the pressure perturbation: $\tilde{P}_{nk} \sim \tau^{-1}$, $\tau \to \infty$, $\tilde{P}_{nk} \sim \tau^{-(2n+3)/3}$, $\tau \to 0$. According to (2.5), for the velocity field generated by the perturbed flow (1.1), we obtain

$$(\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_\varphi) = (m^2\Phi_\rho, (m\rho)^{-2}\Phi_\theta, (m\rho\sin\theta)^{-2}\Phi_\varphi),$$

whence we have for a single harmonic

$$(\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_\varphi)_{nk} \sim (\mu_1(\tau), \tau^{-4/3}\mu_2(\tau), \tau^{-4/3}\mu_3(\tau)), \; \tau \to \infty,$$

where $\mu_j(\tau)$ ($j$ = 1, 2, 3) are bounded functions. For compression

$$(\tilde{u}_r,\ \tilde{u}_\theta,\ \tilde{u}_\varphi)_{nk} \sim \tau^{-2n/3}(\nu_1(\tau),\ \tau^{-4/3}\nu_2(\tau),\ \tau^{-4/3}\nu_3(\tau)),$$

where $\nu_j(\tau)$ are bounded functions for $\tau \to 0$.

These conclusions differ from the results of [1]. The point is that near the critical point $\tau = 0$ (or $\tau = \infty$) the ratio $h/\tau$ (or $\tau/h$) is not small for a thin (or thick) layer, and the expansion in the small parameter $h/r_{10}^3$ (or $r_{10}^3/h$) in the perturbed motion, used for simplifying the problem analysis in [1], is incorrect. The expressions obtained in [1] for $R_{n0}$ are correct only at the initial stage of evolution of boundary perturbations of a thin (thick) layer moving inertially.

Let $\Phi_0 \in W_2^1(\Omega)$, $\boldsymbol{l}\cdot\mathbf{n} \in W_2^{1/2}(\Gamma)$; from the "energy integral" [6] one can then evaluate estimates without solving problem (2.6), (2.9), (2.10), (2.4):

$$\int_\Gamma \Phi_t^2 d\Gamma \sim \tau^{-4/3},\quad \int_\Omega \Phi_\rho^2 d\Omega \sim \tau^{-2/3},\quad \tau \to \infty. \tag{3.13}$$

We recall that $\tau = r_{10}^3 + \gamma$, and $\gamma(t)$ is determined by (1.10), where one must put $\sigma_1 = \sigma_2 = 0$. Thus, if the integral quantities (3.13) are chosen as a measure of stability, the layer motion during expansion must be considered stable.

4. Collapse under the Action of Capillary Forces. Inside the layer the function $\Phi(\rho,\ \theta,\ \varphi,\ t)$ is represented in the form of a series (3.1), where $A_{nk}(\rho,\ t) = A(\rho,\ t)$ are given by equality (3.3). The difference consists of the boundary conditions (2.9), (2.10). Putting $A(r_{10},\ \tau) = A_1(\tau)$, $A(r_{20},\ \tau) = A_2(\tau)$, we obtain a system of form (3.5), (3.6), where one must replace the expressions

$$m_1^2/m_{1tt},\ m_2^2/m_{2tt} \text{ by } \{m_1^{-2}m_{1tt} + r_{10}^{-3}m_1^{-4}\rho_0^{-1}\sigma_1[2 - n(n+1)]\}^{-1}$$
$$\text{and } \{m_2^{-2}m_{2tt} + r_{20}^{-3}m_2^{-4}\rho_0^{-1}\sigma_2[n(n+1) - 2]\}^{-1}$$

respectively. Besides, function $g(\tau)$ is here

$$g^2(\tau) = \frac{6}{\rho_0}\frac{D - 3\sigma_1\tau^{2/3} - 3\sigma_2(h+\tau)^{2/3}}{\tau^{-1/3} - (h+\tau)^{-1/3}}.$$

A study of the system obtained reveals that for $\tau \to 0$ ($\gamma \to -r_{10}^3$) the principal terms of the asymptotic functions $A_1$, $A_2$ coincide for any fixed n with the principal terms of asymptotic system (3.5), (3.6). While the internal surface is unstable, for sufficiently high harmonics with $n \gg \tau^{-1/3}$ the capillary forces restrict the growth of perturbations: $|R_{nk}| < \infty$.

Thus, surface tension lowers the instability somewhat, without removing it completely. A similar effect occurs in the case of collapse of a spherical cavity [7] and a rotating ring [8]. In these examples, obviously, the instability is related, according to the terminology of [9], to a global singularity formed during the process of collapse motion, when the topology of the flow region $\Omega_t$ is destroyed after a finite time.

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# DEVELOPMENT OF THREE-DIMENSIONAL PERTURBATIONS IN RAYLEIGH–TAYLOR INSTABILITIES

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The study of the Rayleigh–Taylor instability (RTI) is a quite relevant problem. Besides the theoretical interest, it is valuable for a number of important practical problems, such as stability studies of shell compression in problems of laser thermonuclear synthesis, obtaining superstrong magnetic fields, etc.

The following clearly expressed stages can be traced in the evolution of RTI: linear, intermediate, regular asymptotic, and turbulent [1, 2]. The linear stage is characterized by a small amplitude $a$ in comparison with the perturbation wavelength L and an exponential velocity growth. When the perturbation amplitude $a$ reaches 0.4L, the process evolves to a stage intermediate between the linear and the regular asymptotic one. At the regular asymptotic stage, starting at $a \approx 0.75L$, heavy fluid "peaks" are definitely formed, breaking down with constant acceleration, as well as light fluid "bubbles," floating with constant velocity. This RTI is unstable [1, 2] and changes into the turbulent stage, during which there is intense interaction of various wavelength perturbations and fluid mixing.

The RTI was investigated in most detail for a planar surface section and a ratio of heavy to light fluid densities tending to infinity. The linear stage was studied in classical papers [3-5], the regular asymptotic stage in [6-8], a phenomenological theory of the turbulent stage was developed in [9], and a discussion of the mechanism of its formation was given in [2].

An analytical mathematical apparatus for analysis is hardly available, however, since experimental studies of RTI are quite difficult. The most complete information can be obtained from numerical calculations; thus, the case of a free surface was investigated in [10], that of two incompressible fluids in [11], and that of two compressible media in [12]. We also point out [13], where numerical calculations of RTI of a compressible shell were performed.

So far only the two-dimensional case was considered both analytically and computationally. The two-dimensional model is, however, physically inadequate: in a physical experiment the two-dimensional structures are destroyed by the transverse shortwave instability, transforming into three-dimensional ones.

The numerical methods used in [10, 11] can be extended, in principle, to the three-dimensional case. This leads, however, to a quite significant increase in the computing time and an increase in the required computer memory, so that detailed calculations cannot be realized on contemporary computers.

In the present paper we perform a numerical experiment on three-dimensional RTI by means of the coarse particle method (see, e.g., [14]), widely recommended in solving a wide class of complex problems of gas hydrodynamics (see, e.g., [14, 15]). The development of two-dimensional RTI up to large amplitudes, when the process becomes substantially nonlinear, was first investigated by the given method in [12].

The full spatial three-dimensional nonstationary system of vortex Euler equations with account of a gravitational field is solved by the coarse particle method

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 117-122, January-February, 1981. Original article submitted September 17, 1979.

0021-8944/81/2201-0097$07.50  